Thus the principal directions of Type IV are indeterminate, and (8) define an invariant function of position

$$\theta = g^{s_1t_1} g^{s_2t_2} g^{s_3t_3} G_{s_1s_2,s_3s} G_{t_1t_2,t_3t} \frac{dx_s}{ds}\frac{dx_t}{ds} = 3\frac{k^2}{x_1^6}.$$

Substituting this value for θ in (1), we obtain, after reduction, the following equations for principal directions:—

$$\phi dx_1 = 9\frac{k^2}{x_1^8}\left(15\frac{k}{x_1} - 14\right)dx_1, \qquad \phi dx_2 = 18\frac{k^2}{x_1^8}\left(1 - \frac{k}{x_1}\right)dx_2,$$

$$\phi dx_3 = 18\frac{k^2}{x_1^8}\left(1 - \frac{k}{x_1}\right)dx_3, \qquad \phi dx_4 = 9\frac{k^3}{x_1^9}dx_4.$$

These equations determine the following directions:—

(i) the parametric lines of x_1, $(dx_2 = dx_3 = dx_4 = 0)$;

(ii) any direction making $dx_1 = dx_4 = 0$;

(iii) the parametric lines of x_4, $(dx_1 = dx_2 = dx_3 = 0)$.

It might be said that these principal directions illustrate both the radial and the stationary characters of the field.

FIELDS OF PARALLEL VECTORS IN THE GEOMETRY OF PATHS

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Communicated May 6, 1922

1. In a former-paper (these PROCEEDINGS, Feb. 1922) Professor Veblen and the writer considered the geometry of a general space from the point of view of the paths in such a space—the paths being a generalization of straight lines in euclidean space. From this point of view it is natural to think of the tangents to a path as being parallel to one another. In this way our ideas may be coördinated with those of Weyl and Eddington who have considered parallelism to be fundamental rather than the paths which we so consider. It is the purpose of this note to determine the geometries which possess one or more fields of parallel vectors, which accordingly define a significant direction, or directions, at each point of the space.

2. The equations of the paths are taken in the form

$$\frac{d^2x^i}{ds^2} + \Gamma^i_{\alpha\beta}\frac{dx^\alpha}{ds}\frac{dx^\beta}{ds} = 0, \tag{2.1}$$

where x^i $(i = 1, \ldots n)$ are the coördinates of a point of a path expressed as functions of a parameter s; $\Gamma^i_{\alpha\beta}$ are functions of the x's such that $\Gamma^i_{\alpha\beta} = \Gamma^i_{\beta\alpha}$.

The components dx^i/ds of the vector tangent to a path being contravariant we put $dx^i/ds = A^i$. In the former paper we observed that the theory of covariant differentiation can be generalized to the geometry of paths by replacing the Christoffel symbols $\left\{ {jk \atop i} \right\}$ by Γ^i_{jk}. Thus the quantities

$$A^i_j = \frac{\partial A^i}{\partial x^j} + \Gamma^i_{\alpha j} A^\alpha \tag{2.2}$$

are the covariant derivatives of A^i; they are the components of a mixed tensor of the second order. Thus $A^i_j\, dx^j$ expresses in invariant form the first variation of the components of A^i as the x's vary. Hence if we write (2.1) in the form

$$A^i_j\, dx^j/ds = 0, \tag{2.3}$$

we see that the first variation of the components of the tangent vector to a path is equal to zero. In this sense we speak of the tangents to a path as parallel.

Suppose now that A^i are the components of any contravariant vector whatever, and consider the vectors at points of any curve C not necessarily a path. The components A^i and coördinates x^i along C are expressible in terms of a parameter s, and dx^i/ds are the components of the tangent to C. If these functions are such that equations (2.3) are satisfied, we say that the vectors A^i are *parallel to one another with respect to the curve.* In particular the tangents to a path are parallel with respect to it. Some time ago Professor Veblen, in discussing the covariant derivative of a tensor, pointed out that it should be interpreted as the system of turning components of the given tensor with respect to the given direction. In this sense (2.3) expresses the fact that the turning components of the vector along the curve are zero.

In order that our definition may be such that if A^i are the components of parallel vectors with respect to a curve so also are $\varphi\, A^i$, where φ is a scalar, we say that the vectors of components A^i are parallel with respect to a curve whose tangents have the components dx^j/ds, provided that a scalar function φ exists such that

$$\left(\frac{\partial A^i}{\partial x^j} + \Gamma^i_{\alpha j} A^\alpha - \frac{\partial \log \varphi}{\partial x^j} A^i\right) \frac{dx^j}{ds} = 0. \tag{2.4}$$

3. Equation (2.4) is satisfied independently of the curve, if

$$\frac{\partial A^i}{\partial x^j} + \Gamma^i_{\alpha j} A^\alpha - \frac{\partial \log \varphi}{\partial x^j} A^i = 0, \tag{3.1}$$

that is

$$A^i_j - \frac{\partial \log \varphi}{\partial x^j} A^i = 0. \tag{3.2}$$

Consequently whenever there exists a field of vectors satisfying (3.1), all the vectors are parallel to one another for any curve, and thus there is a significant direction at each point of the space.

For the second covariant derivatives of any contravariant vector A^i we have the identity

$$A^i_{jk} - A^i_{kj} = -A^\alpha B^i_{\alpha jk}, \tag{3.3}$$

where

$$B^i_{\alpha jk} = \frac{\partial \Gamma^i_{\alpha k}}{\partial x^j} - \frac{\partial \Gamma^i_{\alpha j}}{\partial x^k} + \Gamma^\beta_{\alpha k}\Gamma^i_{\beta j} - \Gamma^\beta_{\alpha j}\Gamma^i_{\beta k}, \tag{3.4}$$

that is the B's are the components of the curvature tensor, as defined in the former paper. From (3.3) it follows that the conditions of integrability of (3.1) are

$$A^\alpha B^i_{\alpha jk} = 0. \tag{3.5}$$

From this equation it follows that a necessary and sufficient condition that (3.1) be completely integrable, that is that there exists a field of vectors parallel to any given vector is $B^i_{\alpha jk} = 0$. From the results of the former paper it follows that in this case the space is euclidean.

If the space is not euclidean, a necessary condition that the A's given by (3.5) shall satisfy (3.2) is

$$A^\alpha B^i_{\alpha jkl} = 0, \tag{3.6}$$

where $B^i_{\alpha jkl}$ is the covariant derivative of $B^i_{\alpha jk}$.

Suppose now that the rank of the matrix of equations (3.5) is such that these equations admit a set of solutions A^i determined to within a scalar factor, and that these functions satisfy (3.6). Differentiating (3.5) covariantly with respect to x^l and taking account of (3.6), we have that the functions A^α_l must satisfy (3.5). In consequence of the above assumptions, it follows that

$$A^\alpha_l = A^\alpha \varphi_l \tag{3.7}$$

where φ_l is a covariant vector. Substituting in (3.3) and making use of (3.5), we find that φ_l is a gradient, and consequently (3.7) is of the form (3.2).

The case when equations (3.5) admit m $(<n)$ sets of solutions, in terms of which any set of solutions is linearly expressible can be handled by a method similar to that used in §7 of the former paper. In this case any vector at a point P in the m-fold bundle of vectors determined by the m vectors at P is parallel to a vector in the corresponding bundle at any other point of the space.

4. In the preceding section we have given the conditions for one or more fields of vectors in invariantive form. Now we shall show how all such fields may be obtained by making a suitable choice of coördinates.

Suppose we have m ($<n$) fields of parallel vectors of components $A^i_{(p)}$, where $i = 1, \cdots n$; $p = 1, \cdots m$. If $B^i_{(p)}$ denote the components in another set of variables y^i, we have

$$B^i_{(p)} = A^\alpha_{(p)} \frac{\partial y^i}{\partial x^\alpha}. \tag{4.1}$$

If we show that there exist n independent functions y^i such that

$$X_p (y^i) \equiv A^\alpha_{(p)} \frac{\partial y^i}{\partial x^\alpha} = 0 \quad (i \neq p), \tag{4.2}$$

then in the new set of coördinates all the components of the vectors will be zero except those of the form $B^p_{(p)}$.

If we form the Poisson operator for (4.2), namely $(X_p X_q - X_q X_p)(y^i)$, we have in consequence of (3.1)

$$(X_p X_q - X_q X_p) (y^i) = A^\alpha_{(p)} \frac{\partial \log \varphi_q}{\partial x^\alpha} X_q (y^i) - A^\alpha_{(q)} \frac{\partial \log \varphi_p}{\partial x^\alpha} X_p (y^i),$$

where φ_p ($p = 1, \ldots m$) are the functions φ appearing in equations of the form (3.1). Hence (4.2) is a complete system.

If we let p take all the values from 1 to m, there are in accordance with the theory of complete systems $n - m$ independent solutions, which we take for $y^{m+1}, \ldots, y^n$. If we exclude from the system (4.2) the equation $X_r = 0$ where r has a value from 1 to m, we have a complete system of $m - 1$ equations, of which $n - m$ independent solutions are $y^{m+1}, \ldots, \ldots y^n$, and the other we take for y^r. Hence if there exist m fields of parallel vectors, the coördinates can be chosen so that all the components are zero except those of theform $A^p_{(p)} (p = 1, \ldots, m)$, and consequently our problem reduces to the determination of geometries for which equations (3.1) admit solutions of this kind.

5. In order that equations (3.1) admit solutions $A^1_{(1)} \neq 0$, $A^i_{(1)} = 0$ $(i \neq 1)$, we must have

$$\Gamma^1_{1j} = 0, \quad \Gamma^i_{1j} = \frac{\partial}{\partial x^j} \log \psi \quad \begin{pmatrix} i = 2, \ldots n \\ j = 1, \ldots n \end{pmatrix}, \tag{5.1}$$

where ψ is an arbitrary function of the x's, and then $A^1_{(1)} = \varphi/\psi$. Consequently if we choose the Γ's with one or two subscripts 1 as given by (5.1), and take the others as arbitrary functions of the x's, we have the most general geometry with one field of parallel contravariant vectors.

In like manner any geometry with m fields of parallel contravariant vectors can be obtained by choosing

$$\Gamma^i_{pj} = 0 \quad \Gamma^p_{pj} = \frac{\partial}{\partial x^j} \log \psi_p \quad \begin{pmatrix} p = 1, \ldots m \\ i \neq p \\ j = 1, \ldots n \end{pmatrix}, \tag{5.2}$$

where ψ_p is an arbitrary function of $x^p, x^{m+1}, \ldots, x^n$.

6. Suppose now that the geometry is Riemannian, the fundamental form being

$$ds^2 = g_{ij}\, dx^i\, dx^j \quad (g_{ij} = g_{ji}) \tag{6.1}$$

Since

$$\Gamma^i_{1j} = \frac{1}{2} g^{i\alpha} \left(\frac{\partial g_{1\alpha}}{\partial x^j} + \frac{\partial g_{\alpha j}}{\partial x^1} - \frac{\partial g_{1j}}{\partial x^\alpha}\right),$$

equations (5.1) are in this case equivalent to

$$\frac{\partial g_{1\alpha}}{\partial x^j} + \frac{\partial g_{\alpha j}}{\partial x^1} - \frac{\partial g_{1j}}{\partial x^\alpha} = 2 g_{1\alpha} \frac{\partial}{\partial x^j} \log \psi \tag{6.2}$$

for $\alpha, j = 1, \ldots n$. When we take $\alpha = j = 1$, we find $g_{11} = \psi^2 \Phi(x^2, \ldots x^n)$. When we take $j = 1$, $\alpha \neq 1$, we find

$$\frac{\partial}{\partial x^1}\left(\frac{g_{1\alpha}}{\psi}\right) = \Phi \frac{\partial \psi}{\partial x^\alpha} + \frac{\psi}{2} \frac{\partial \Phi}{\partial x^\alpha}. \tag{6.3}$$

Consequently $g_{1\alpha}$ are given by quadratures, and likewise $g_{\alpha j}$ ($\alpha \neq 1$, $j \neq 1$) from (6.2).

In particular if ψ = const., by interchanging α and j in (6.2), we find that $\partial g_{\alpha j}/\partial x^1 = 0$ for all values of α and j. Moreover (6.2) becomes

$$\frac{\partial g_{1\alpha}}{\partial x^j} = \frac{\partial g_{1j}}{\partial x^\alpha}. \tag{6.4}$$

As a first consequence of this equation we have that g_{11} is a constant which may be taken equal to unity. Again (6.4) are the necessary and sufficient conditions that

$$g_{1\alpha}\, dx^\alpha = dx^1 + d\varphi\, (x^2, \ldots, x^n).$$

If then x^1 is replaced by $x^1 - \varphi\, (x^2, \ldots x^n)$, the form (6.1) becomes

$$ds^2 = dx_1^2 + g_{ij}\, dx^i\, dx^j \quad (i, j = 2, \ldots n), \tag{6.5}$$

where g_{ij} are independent of x^1. A space with linear element (6.5) is the most general which admits a translation into itself. (Bianchi, *Teoria dei gruppi continui*, Pisa, 1918, p. 500.) The space-time manifold of four dimensions used by Einstein in his cosmological considerations is of the type (6.5), x^1 being the coördinate of time.

In the case of $m (< n)$ fields of parallel vectors for which $\Gamma^i_{pj} = 0$, ($p = 1, \ldots m$; $i, j = 1, \ldots, n$), these equations are equivalent to

$$\frac{\partial g_{ji}}{\partial x^p} = 0, \quad \frac{\partial g_{pi}}{\partial x^j} = \frac{\partial g_{pj}}{\partial x^i}. \tag{6.6}$$

From the first of these it follows that all of the functions g_{ij} are independent of $x^1, \ldots x^m$. From the second of (6.6) we find that g_{pq} ($pq = 1$,

..m) are constants. Then as in the above case we show that the linear element can be put in the form

$$ds^2 = (dx^1)^2 + \ldots + (dx^m)^2 + g_{ij}\, dx^i\, dx^j \quad (i, j = m+1, \ldots n),$$

where g_{ij} are independent of $x^1, \ldots, x^m$. When $m = n - 1$, equation (6.7) is reducible to the euclidean form.

7. If A^i are the contravariant components of a vector in a Riemannian geometry, its covariant components A_i, are given by

$$A^i = g^{i\alpha} A_\alpha. \tag{7.1}$$

When this expression is substituted in (3.1), we obtain

$$\frac{\partial A_i}{\partial x^j} - \Gamma^\alpha_{ij} A_\alpha - A_i \frac{\partial \log \varphi}{\partial x^j} = 0. \tag{7.2}$$

When we are dealing with a non-Riemannian geometry we say that a field of parallel covariant vectors is one which satisfies (7.2).

The conditions of integrability of (7.2) are

$$A_\alpha B^\alpha_{ijk} = 0. \tag{7.3}$$

In order that the A's given by (7.3) shall satisfy (7.2) it is necessary that

$$A_\alpha B^\alpha_{ijkl} = 0. \tag{7.4}$$

As in the case of contravariant vectors, it can be shown that when there are m ($< n$) independent sets of solutions of (7.3) which satisfy (7.4) there exist m fields of parallel, covariant vectors.

The methods of §§ 4, 5 cannot be applied to the case of covariant fields.

ON THE INFLUENCE OF DENSITY OF POPULATION UPON THE RATE OF REPRODUCTION IN DROSOPHILA[1]

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Communicated, May 5, 1922

It has long been known that degree of crowding of organisms in a given space, or the density of the population, has an influence upon various vital processes of the individuals composing the population. In the matter of growth Semper[2] and before him Jabez Hogg[3] showed that volume of water apart from food and other conditions has an influence upon the rate. This subject has again been studied recently by Bilski[4]. Farr[5] showed that there is in man a definite relation between density of population and the death rate. This old work of Farr's has recently been gone over carefully and confirmed by Brownlee.[6] Drzwina and Bohn[7] show that a particular concentration of a toxic substance, just lethal for a single